toms also showed less favorable comparison between measured and simulated data at the higher frequencies. It has been speculated that this could be due to higher cross-talk among the quadrants [10] or a change in the current pattern on the dipoles [11] at the high frequencies. Nonetheless, the measured data still follows the general pattern of the FDTD data.

V. Discussion

We have presented comparisons between measured data from an inhomogeneous phantom and those predicted by an FDTD program used in treatment planning for deep regional hyperthermia. These were limited to the case of equal amplitudes and phases at three frequencies. A more extensive range of comparisons for different configurations of amplitude and phase would be desirable, but the number of measurements is hindered by time-consuming logistical problems. The Utah phantom itself requires almost two days to construct and suffers from a short "shelf life" due to the materials used and boundary diffusion. However, this phantom represents a substantial improvement over conventional homogeneous phantoms in testing the ability of simulation methods to account for the inhomogeneities of the human body. In particular, it is hoped that the results presented here will promote confidence in the treatment planning program using the FDTD method.

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Alternative Field Representations and Integral Equations for Modeling Inhomogeneous Dielectrics

John L. Volakis

***Abstract*—New volume and volume-surface integral equations are presented for modeling inhomogeneous dielectric regions. In particular, it is shown that materials with non-trivial permeability and permittivity can be modeled using a single unknown equivalent current or field component. The presented integral equations result in more efficient numerical implementations and should therefore be useful in a variety of electromagnetic applications.**

I. Introduction

The modeling of inhomogeneous dielectrics via an integral equation approach is traditionally accomplished via the introduction of equivalent volume electric and magnetic currents [1]–[8]. For a dielectric with non-trivial permittivity and permeability this type of modeling implies six scalar unknowns at each volume location. As a result, the implementation of the resulting integral equation is computationally intensive and has excessive storage requirements.

In this paper it is demonstrated that any inhomogeneous dielectric material, regardless of its permittivity and permeability profile, can be modeled by a single electric or magnetic current density. Alternatively, either the electric or magnetic fields within the dielectric can be used as the unknown quantities. It appears though that one must pay a price for resorting to these reduced-unknown and/or kernal-singularity representations. Specifically, because they involve derivatives of the unknown quantities, a higher (at least linear) basis function is required for discretizing the resulting integral equations. However, it is possible to relax this requirement by resorting to a new volume-surface field representation. In this case, the undifferentiated electric or magnetic field within the dielectric is the unknown quantity along with the corresponding tangential electric or magnetic fields on the outer boundary. Provided the dielectric volume is not composed of a single thin layer, this volume-surface integral equation still represents a nearly fifty percent reduction in the number of unknowns relative to traditional implementations.

II. Volume Representations

Let us consider the dielectric/ferrite volume V_d, shown in Fig. 1, having relative constitutive parameters ϵ_r and μ_r which are arbitrary functions of position. Assuming some exterior excitation, $(\boldsymbol{E}^i, \boldsymbol{H}^i)$, the total field can be written as

$$\boldsymbol{E} = \boldsymbol{E}^i + \boldsymbol{E}^s \quad \boldsymbol{H} = \boldsymbol{H}^i + \boldsymbol{H}^s \tag{1}$$

where $(\boldsymbol{E}^s, \boldsymbol{H}^s)$ are the scattered fields caused by the presence of the dielectric. Traditionally [1] the scattered fields are formulated

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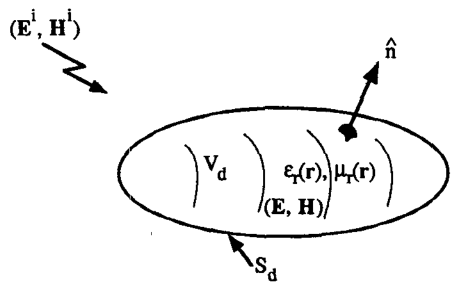

Fig. 1. Illustration of the inhomogeneous dielectric volume V_d enclosed by the surface S_d.

in terms of the equivalent currents

$$J_{eq} = jk_o Y_o(\epsilon_r - 1)E, \quad M_{eq} = jk_o Z_o(\mu_r - 1)H \tag{2}$$

with k_o and $Z_o = 1/Y_o$ being the free space wavenumber and intrinsic impedance, respectively. In terms of these effective or equivalent current densities, the scattered field is given by

$$E^s = \iiint_{V_d} [\nabla \times \bar{\Gamma}_o(r, r') \cdot M_{eq}(r') + jk_o Z_o \bar{\Gamma}_o(r, r') \cdot J_{eq}(r')]\, dv' \tag{3}$$

in which r and r' denote the observation and integration points, respectively,

$$\bar{\Gamma}_o(r, r') = -\left[\bar{I} + \frac{\nabla\nabla}{k_o^2}\right] G_o(r, r'), \tag{4}$$

is the free space dyadic Green's function,

$$\nabla \times \bar{\Gamma}_o(r, r') = -\nabla G_o(r, r') \times \bar{I}, \tag{5}$$

$$G_o(r\, r') = \frac{e^{-jk_o|r-r'|}}{4\pi|r - r'|},$$
$$\bar{I} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} \tag{6}$$

is the unit dyad and H^s is given by the dual of (3). By substituting (3) and its dual into (1) and then into (2), we obtain the coupled set of integral equations:

$$\frac{J_{eq}(r)}{jk_o Y_o(\epsilon_r - 1)} - E^s = E^i \qquad r \in V_d \tag{7a}$$

$$\frac{M_{eq}(r)}{jk_o Z_o(\mu_r - 1)} - H^s = H^i \qquad r \in V_d \tag{7b}$$

for a solution of the equivalent currents J_{eq} and M_{eq}.

The aforementioned formulation appears to be the only approach that has so far been utilized for three dimensional implementations. However, as noted in the literature [4], [5], [9], the singularity of the kernal in (3) presents numerical difficulties. Also, for non-trivial permeability, six scalar unknowns are involved in the solution of (7). One way to alleviate the first of these difficulties is by resorting to higher order basis functions and expressing, for example, E^s as

$$E^s = \iiint_{V_d} \Big\{ M_{eq} \times \nabla G_o(r, r') - jk_o Z_o J_{eq} G_o(r, r') - \frac{jZ_o}{k_o} \nabla' \cdot J_{eq}(r') \nabla G_o(r, r') \Big\}\, dv'. \tag{8}$$

In this, ∇' implies differentiation with respect to the primed/integration coordinates and we remark that such an expression and its dual can be considered as the volume equivalent to the Stratton-Chu surface integral equations.

Although the above approach appears to be the most popular in modeling three-dimensional dielectrics, it can be shown that there are several other ways to formulate the problem. Most importantly, it can also be shown that (7) can be replaced with an equivalent system which involves only three (not six) scalar unknowns. Specifically, from Maxwell's equations [10] the radiation of M_{eq} is indistinguishable from the radiation of the electric current:

$$J'_{eq} = \frac{\nabla \times M_{eq}}{jk_o Z_o} \tag{9}$$

This can be combined with (2) giving a single equivalent electric current

$$J''_{eq} = jk_o Y_o(\epsilon_r - 1)E + \nabla [(\mu - 1)H]$$
$$= \frac{(\epsilon_r - 1)}{\epsilon_r} \nabla \times H + \nabla \times [(\mu_r - 1)H] \tag{10}$$

for representing the scattered fields (E^s, H^s). From the dual of (3) we then obtain that the scattered magnetic field due to the current density (10) is

$$H^s = \iiint_{V_d} [\nabla G_o(r, r') \times \bar{I}] \cdot \Big\{ \frac{\epsilon_r(r') - 1}{\epsilon_r(r')} \nabla' \times H(r') + \nabla' \times [(\mu_r(r') - 1)H(r')] \Big\}\, dv'. \tag{11}$$

When this is used in (1) we deduce the integral equation

$$H'(r) = H(r) - \iiint_{V_d} [\nabla G_o(r, r') \times \bar{I}] \cdot \Big\{ \frac{\epsilon_r(r') - 1}{\epsilon_r(r)} \nabla' \times H(r') + \nabla' \times [(\mu_r(r') - 1)H(r')] \Big\}\, dv' \qquad r \in V_d \tag{12}$$

where the unknown quantity is now the magnetic field within V_d. Using a similar procedure it can be also shown that the scattered field may instead be represented by the radiation of a single magnetic current density:

$$M''_{eq} = -\frac{(\mu_r - 1)}{\mu_r} \nabla \times E - \nabla \times [(\epsilon_r - 1)E] \tag{13}$$

From the first of (1) and (3), we then deduce the integral equation

$$E'(r) = E(r) \iiint_{V_d} [\nabla G_o(r, r') \times \bar{I}] \cdot \Big\{ \frac{\mu_r(r') - 1}{\mu_r(r')} \nabla' \times E(r') + \nabla' \times [(\epsilon_r(r') - 1)E(r')] \Big\}\, dv' \tag{14}$$

which as expected is the dual of (12). We observe that the kernel singularity associated with (12) and (14) is the same as that associated with (8). In addition, as in the case of the integral equation (7) in conjunction with (8), linear expansion functions such as those in [3] or [4] are required for the discretization of (12) and (14). Thus, even though the new integral equations (12) and (14) have half the unknowns, this was not achieved at the expense of increasing the kernel's singularity or the order of the expansion basis required in their implementation. It is remarked that special forms of these integral equations have already been successfully implemented for two dimensional applications [11], [12].

III. Volume-Surface Representation

The requirement to employ linear basis in connection with the implementation of (12) and (14) can be relaxed by resorting to a volume-surface integral equation (VSIE) such as that derived in [13] and [14] for two dimensional simulations. To do so we begin with (3) which in conjunction with (2) can be rewritten as

$$E^s = E_e^s + E_m^s = -k_o^2 \iiint_{V_d} [\epsilon_r(r') - 1] E(r') \cdot \overline{\Gamma}_o(r, r') - jk_o Z_o \nabla \times \iiint_{V_d} [\mu_r(r') - 1] H(r') G_o(r, r')\, dv' \tag{15}$$

where E_m^s is associated with the second integral and represents the field due to the magnetic equivalent current defined in (2). Setting $H = \nabla \times E/jk_o Z_o \mu_r$ in this integral, and invoking the identities

$$\nabla \times [\nabla' \times \phi E] = \nabla \times [\nabla' \phi \times E] + \nabla \times [\phi \nabla' \times E]$$
$$\nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi$$

we obtain

$$E_m^s = F_m^{s1} + F_m^{s2} + F_m^{s3} \tag{16}$$

with

$$F_m^{s1} = \nabla \times \iiint_{V_d} \nabla' \times \left\{ \left(1 - \frac{1}{\mu_r(r')}\right) G_o(r, r') E(r') \right\} dv' \tag{17}$$

$$F_m^{s2} = -\nabla \times \iiint_{V_d} \left\{ \left(1 - \frac{1}{\mu_r(r')}\right) \nabla' G_o(r, r') \times E(r') \right\} dv' \tag{18}$$

$$F_m^{s3} = \nabla \times \iiint_{V_d} \left\{ G_o(r, r') \nabla' \left(\frac{1}{\mu_r(r')}\right) \times E(r') \right\} dv' \tag{19}$$

These integral expressions can be simplified through the use of various integral and differential identities.

The volume integral in (17) can be transformed to a surface integral by invoking Stoke's identity:

$$\iiint_{V_d} (\nabla' \times A)\, dv' = \oiint_{S_d} (\hat{n}' \times A)\, ds' \tag{20}$$

where S_d is the surface enclosing V_d and $\hat{n}' = \hat{n}(r')$ denotes the outward unit normal to the surface S_d. We have

$$F_m^{s1} = \nabla \times \oiint_{S_d} \left(1 - \frac{1}{\mu_r(r')}\right) G_o(r, r') [\hat{n}' \times E(r')]\, ds' = -\oiint_{S_d} \left(1 - \frac{1}{\mu_r(r')}\right) [\hat{n}' \times E(r')] \times \nabla G_o(r, r')\, ds' \tag{21}$$

which is an integral involving the undifferentiated tangential electric field over the surface enclosing V_d. Turning now to the integral in (18) we first rewrite it as

$$F_m^{s2} = -\iiint_{V_d} \left[1 - \frac{1}{\mu_r(r')}\right] \nabla \times [\nabla' G_o(r, r') \times E(r')]\, dv' \tag{22}$$

and we note that [15, p. 487]

$$\nabla \times [\nabla' G_o \times E(r')] = E(r') \nabla^2 G_o - E(r') \cdot \nabla\nabla G_o. \tag{23}$$

Then, upon invoking the differential equation

$$\nabla^2 G_o(\mathbf{r}, \mathbf{r}') + k_o^2 G_o(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \tag{24}$$

where $\delta(r')$ denotes the Dirac delta function, it follows that

$$F_m^{s2} = -k_o^2 \iiint_{V_d} \left[1 - \frac{1}{\mu_r(r')}\right] E(r') \cdot \overline{\Gamma}_o(r, r')\, dv' + \left[1 - \frac{1}{\mu_r(r')}\right] E(r'). \tag{25}$$

Again, this involves only the undifferentiated electric field within the dielectric's volume. Finally, the last integral in (16) can be readily simplified and written as

$$F_m^{s3} = \iiint_{V_d} \nabla \times \left\{ G_o(r, r') \nabla' \left[\frac{1}{\mu_r(r')}\right] \times E(r') \right\} dv' = \iiint_{V_d} \nabla G_o(r, r') \times \left\{ \nabla' \left[\frac{1}{\mu_r(r')}\right] \times E(r') \right\} dv'. \tag{26}$$

When (21), (25), and (26) are substituted into (16) and then into (15), we find that the total scattered field can be expressed as

$$\begin{aligned} E^s = & -k_o^2 \iiint_{V_d} \left[\epsilon_r(r') - \frac{1}{\mu_r(r')}\right] E(r') \cdot \overline{\Gamma}_o(r, r')\, dv' \\ & + \iiint_{V_d} \nabla G_o(r, r') \times \left\{ \nabla' \left[\frac{1}{\mu_r(r')}\right] \times E(r') \right\} dv' \\ & - \oiint_{S_d} \left[1 - \frac{1}{\mu_r(r')}\right] [\hat{n}' \times E(r')] \times \nabla G_o(r, r')\, ds' \\ & + \left[1 - \frac{1}{\mu_r(r')}\right] E(r) \end{aligned} \tag{27}$$

For two dimensional simulations where the material parameters and the fields are invarient with respect to z, this expression can be readily shown to reduce to the VSIE given by Jin, etc. [13, equations 28 and 31]. Expression (27) is also similar to the VSIE given by Tai [16]. However, Tai's expression was left in terms of differentiated field quantities and is only applicable to homogeneous dielectrics.

To obtain an integral equation on the basis of (27) we substitute this into the first of (1) and upon taking the principal value of the

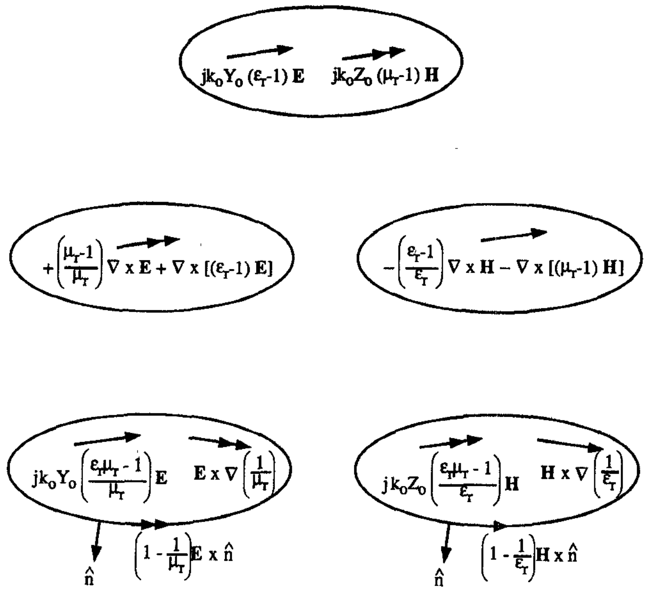

Fig. 2. Different volume equivalent currents for modeling the scattering by the inhomogeneous dielectric volume V_d in Fig. 1.

appropriate integrals we have

$$-k_o^2 \iiint_{V_d - V_o} \left[\epsilon_r(r') - \frac{1}{\mu_r(r')}\right] E(r') \cdot \bar{\Gamma}_o(r, r')\, dv'$$
$$+ \iiint_{V_d} \left\{E(r') \times \nabla' \left[\frac{1}{\mu_r(r')}\right]\right\} \times \nabla G_o(r, r')\, dv'$$
$$- \oiint_{S_d - S_o} \left[1 - \frac{1}{\mu_r(r')}\right] [\hat{n}' \times E(r')] \times \nabla G_o(r, r')\, dv'$$
$$+ E^i = \begin{cases} E(r) & r \text{ not in } V_d \\ \frac{1}{2}\left[1 + \frac{1}{\mu_r(r)}\right] E(r) & r \text{ on } S_d \\ \frac{1}{3}[\epsilon_r(r) + 2\mu_r(r)] E(r) & r \text{ in } V_d. \end{cases} \quad (28)$$

In this, V_o is a vanishingly small spherical volume whereas S_o is a vanishingly small hemispherical surface both having their centers at r. As given, (28) can be discretized via the moment method or some other technique for a solution of $E(r)$ within the dielectric. Its kernal has, of course, the same singularity as (7a) but involves only a single unknown vector field in comparison with the two vector unknowns appearing in (7). If linear rather than pulse basis are employed for the solution of (28), it may then be desirable to rewrite the first integral of (28) in the form given by (8) with $M_{eq} = 0$ and

$$J_{eq} = \frac{jk_o}{Z_o}\left[\frac{\epsilon_r(r)\mu_r(r) - 1}{\mu_r(r)}\right] \mathbf{E}(r) \quad (29)$$

However, in this case one could also resort to the alternative integral equations (12) or (14). Of course, the dual of (28) is another integral equation. Further, linear combinations of (28) and its dual or (12) and (14) can be utilized if so desired.

In closing, we remark that if μ_r and/or ϵ_r are discontinuous within V_d, the surface integral in (27) and its dual must then be replaced by

$$\sum_i \oiint_{S_{d_i}} [u_+^i(r') - u_-^i(r')]\, [\hat{n}_i(r') \times F(r')] \times \nabla G_o(r, r')\, ds'$$

where $F = E$ or H. Here, S_{d_i} denotes the ith boundary separating the regions having different constitutive parameters, $\hat{n}_i(r)$ is the unit normal to S_i pointing from the $-$ side to the $+$ side (outermost side) and $u_\pm^i$ denotes the inverse relative dielectric constant at the $+$ or $-$ side of the surface S_{d_i}. In particular $u^i = 1/\mu_r^i$ for the E-field integral equation (27) and $u^i = 1/\epsilon_r^i$ for the H-field integral equation.

IV. Conclusion

Some alternative formulations were proposed for modeling three-dimensional inhomogeneous dielectrics. These are summarized in Fig. 2 and the aim of the investigation was to generate integral equations for the fields within the dielectric scatterer utilizing the minimum number of unknowns and the least singular kernels. A purely volume integral equation was derived involving half the unknowns required with traditional equations for ferrite materials. The implementation of this reduced-unknown volume equation implies use of (at least) linear basis functions and to relax this requirement a volume-surface integral equation was derived. All of the integral equations presented here appear to be more efficient than the traditional ones without compromising the kernel's singularity and this has already been demonstrated in two-dimensional implementations. They should thus be found useful in a variety of radiation, scattering or SAR applications.

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